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Ergodic theorem for the two-dimensional super-Brownian motion with super-Brownian immigration*

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Abstract By investigating the related nonlinear equation, the ergodic theorem is obtained for the super-Brownian with super-Brownian immigration in two dimension.

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Super -Brownian motion with super-Brownian immigration (SBMSBI) was firstly studied by Hong and Li[1]. They established the (weak) central limit theorem for dimension $d \geqslant 3$, and obtained, as a consequence, the weak ergodic theorem (law of large number). But they did not give the result for $d=2$. In this paper, this case ($d=2$) is discussed, and by investigating the related evolution equation, a weak ergodic theorem for the SBMSBI is obtained.

Let $C(\mathbb{R}^d)$ denote the space of continuous bounded functions on $\mathbb{R}^d$. Fix a constant $p>d$ and let $\phi_p(x):=(1+|x|^2)^{-p/2}$ for $x \in \mathbb{R}^d$ and $C_p(\mathbb{R}^d):=\{f \in C(\mathbb{R}^d): |f(x)| \leqslant \text{const}\cdot\phi_p(x)\}$. Let $M_p(\mathbb{R}^d)$ be the space of Radon measures μ on $\mathbb{R}^d$ such that $\langle\mu, f\rangle := \int f(x)\mu(\mathrm{d}x) < \infty$ for all $f \in C_p(\mathbb{R}^d)$. Endow $M_p(\mathbb{R}^d)$ with the p-vague topology, that is, $\mu_k \to \mu$ if and only if $\langle\mu_k, f\rangle \to \langle\mu, f\rangle$ for all $f \in C_p(\mathbb{R}^d)$. Throughout the paper, λ denotes the Lebesgue measure on $\mathbb{R}^d$.

Suppose that $W=(w_t,\ t \geqslant 0)$ is a standard Brownian motion in $\mathbb{R}^d$ with semigroup $(P_t)_{t \geqslant 0}$. Firstly recall the SBMSBI briefly (see ref. [1]). Let $\gamma:=\{\gamma_t, t \geqslant 0\}$ be a continuous $M_p(\mathbb{R}^d)$-valued function, $X^\gamma:=\{X_t^\gamma,\ t \geqslant 0,\ P_\lambda^\gamma\}$ the super-Brownian motion with immigration rate γ, is given by the Laplace functional[2,3]:

$$P_\lambda^\gamma \exp(-\langle X_t^\gamma, f\rangle) = \exp(-\langle\lambda, v(t,\cdot)\rangle - \int_0^t \langle\gamma_s,\ v(t-s,\cdot)\rangle \mathrm{d}s),\ f \in C_p^+(\mathbb{R}^d), \tag{1}$$

where $v(\cdot,\cdot)$ is the unique positive mild solution of the evolution equation

$$\begin{cases} \dot{v}(t) = \Delta v(t) - v^2(t), \\ v(0) = f. \end{cases} \tag{2}$$

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The so-called SBMSBI is the super-Brownian motion with random immigration, where the immigration rate is determined by the trajectory of another super-Brownian motion. For details, one can construct a probability space $(\Omega, \mathscr{F}, Q)$ on which the processes $\{\rho_t : t \geqslant 0\}$ and $\{X_t^\rho : t \geqslant 0\}$ are defined, where $\{\rho_t : t \geqslant 0, P_\lambda\}$ is a super-Brownian motion with $\rho_0 = \lambda$ and, giving the trajectory $\{\rho_t : t \geqslant 0\}$ for P_λ-a.s., the process $\{X_t^\rho : t \geqslant 0, P_\lambda^\rho\}$ is a super-Brownian motion with immigration determined by $\{\rho_t : t \geqslant 0\}$, which is called SBMSBI with $X_0^\rho = \lambda$ and,

$$Q(\cdot) := P_\lambda\{P_\lambda^\rho(\cdot)\} = \int P_\lambda^\rho(\cdot) P_\lambda(\mathrm{d}\rho).$$

Then from (1) and the representation for the occupation time of super-Brownian motion[4], the Laplace transition functional of the SBMSBI X^ρ under the law Q is

$$Q\exp(-\langle X_t^\rho, f\rangle) = \exp(-\langle \lambda, v(t, \cdot)\rangle - \langle \lambda, u(t, \cdot)\rangle \mathrm{d}s), \; f \in C_p^+(\mathbb{R}^d), \tag{3}$$

where $v(\cdot,\cdot)$ is given by (2) and $u(\cdot,\cdot)$ is the unique positive mild solution of the evolution equation

$$\begin{cases} \dot{u}(t) = \Delta u(t) - u^2(t) + v(t), \\ u(0) = 0. \end{cases} \tag{4}$$

Now, the main result can be obtained.

Theorem 1. *Let $d = 2$, then as $T \to \infty$*

$$T^{-1} X_T^\rho \to \xi \cdot \lambda \text{ weakly (with respect to } Q),$$

where ξ is a non-negative, infinitely divisible random variable whose Laplace transform is given by

$$Q\exp\{-\theta\xi\} = \exp\{-\langle \lambda, w(1, \cdot; \theta)\rangle\}, \tag{5}$$

where $w \equiv w(t, x; \theta)$ is the mild solution of the evolution equation

$$\begin{cases} \dot{w}(t) = \Delta w(t) - w^2(t) + \theta p_t(x), \\ w(0) = 0. \end{cases} \tag{6}$$

Now we proceed to the proof of Theorem 1. Let $v_T(t, x)$ be the mild solution of the equation

$$\begin{cases} \dot{v}(t) = \Delta v(t) - v^2(t), \\ v(0) = T^{-1} f. \end{cases} \tag{7}$$

In the following lemmas, we consider $f \in C_p(\mathbb{R}^d)^+$ such that $\langle \lambda, f\rangle = 1$.

Lemma 1. *Let $d = 2$, then we have*

$$\lim_{T\to\infty} T^2 v_T(Tt,\ T^{1/2}x) = p_t(x),$$

where $p_t(x)$ *is the transition density function of the Brownian motion.*

Proof. In this and the following proofs, C will denote a constant which may take different values in different lines. The mild form of (7) is

$$v_T(t,x) = P_t(T^{-1}f)(x) - \int_0^t P_s v_T^2(t-s,\cdot)(x)\mathrm{d}s.$$

Then

$$v_T(Tt,\ T^{1/2}x) = P_{Tt}(T^{-1}f)(T^{1/2}x) - \int_0^{Tt} P_s v_T^2(Tt-s,\cdot)(T^{1/2}x)\mathrm{d}s, \tag{8}$$

note that $d=2$, we can calculate that as $T\to\infty$,

$$\begin{aligned} T^2 P_{Tt}(T^{-1}f)(T^{1/2}x) &= T\int p(Tt,\ T^{1/2}x, y)f(y)\mathrm{d}y \\ &= T\int (2\pi Tt)^{-1}\exp\left\{-\frac{(T^{1/2}x-y)^2}{2Tt}\right\}f(y)\mathrm{d}y \\ &= \int (2\pi t)^{-1}\exp\left\{-\frac{(x-T^{-1/2}y)^2}{2t}\right\}f(y)\mathrm{d}y \\ &\to p_t(x)\langle\lambda, f\rangle = p_t(x). \end{aligned}$$

On the other hand, for any $f\in C_p(\mathbb{R}^d)^+$ we have

$$\| P_s f\| \leqslant C\cdot(1\wedge s^{-d/2}),$$

it follows that

$$\begin{aligned} &T^2\int_0^{Tt} P_s v_T^2(Tt-s,\cdot)(T^{1/2}x)\mathrm{d}s \\ &\leqslant C\cdot T^2\int_0^{Tt} P_s[P_{Tt-s}T^{-1}f]^2(T^{1/2}x)\mathrm{d}s \\ &\leqslant C\cdot P_{Tt}f(T^{1/2}x)\cdot\int_0^{Tt}(1\wedge(Tt-s)^{-1})\mathrm{d}s \\ &= C\cdot T^{-1}\int_0^{Tt}(1\wedge s^{-1})\mathrm{d}s\int(2\pi t)^{-1}\exp\left\{-\frac{(x-T^{-1/2}y)^2}{2t}\right\}f(y)\mathrm{d}y \\ &\to 0. \end{aligned}$$

Combining with (8), the desired conclusion is reached.

Lemma 2. *Let* $d=2$, $w_T(t,x)$ *be the mild solution of the equation*

$$\begin{cases} \dot{w}_T(t,x) = \Delta w_T(t,x) - w_T^2(t) + T^2 v_T(Tt,\ T^{1/2}x), \\ w_T(0) = 0. \end{cases} \tag{9}$$

Then $w(t,x) := \lim_{T\to\infty} w_T(t,x)$ *exists and, is the mild solution of the equation*

$$\begin{cases} \dot{w}(t,x) = \Delta w(t,x) - w^2(t,x) + p_t(x), \\ w(0) = 0, \end{cases} \tag{10}$$

and the limit taken in $C([0,+\infty),\ L^2(\lambda))$ *and pointwise.*

Proof. The mild form of (9) is

$$w_T(t,x) = \int_0^t P_{t-s}(T^2 v_T(Ts,\ T^{1/2}\cdot))(x)\mathrm{d}s - \int_0^t P_{t-s} w_T^2(s,\cdot)(x)\mathrm{d}s. \tag{11}$$

It can be verified that $\{w_T\}_{T>0}$ is $L^2(\lambda)$-Cauchy. Let $T_1, T_2 > 0$, then from (11), ones have

$$|w_{T_1}(t,x) - w_{T_2}(t,x)|^2 \leqslant 2\left(\int_0^t P_{t-s}[w_{T_2}^2(s,\cdot) - w_{T_1}^2(s,\cdot)](x)\mathrm{d}s\right)^2 + 2\left(\int_0^t P_{t-s}[T_2^2 v_{T_2}(T_2 s,\ T_2^{1/2}\cdot) - T_1^2 v_{T_1}(T_1 s,\ T_1^{1/2}\cdot)](x)\mathrm{d}s\right)^2.$$

But from (8), one can see

$$T^2 v_T(Tt,\ T^{1/2}x) \leqslant T^2 P_t(T^{-1}f)(T^{1/2}x) \leqslant (2\pi t)^{-1}.$$

So the limit in Lemma 1 is taken in $C([0,+\infty),\ L^2(\lambda))$ and pointwise. Combining this, the remaining proof is similar to Proposition 3.9 in ref. [5]: firstly, we can prove that the limit $w(t,x)$ exists in $C([0,+\infty),\ L^2(\lambda))$, then, the limit is taken in pointwise and satisfying (10), finally, the mild solution of (10) is unique. The detail is ommited.

Lemma 3. $\lim_{T\to\infty}\langle\lambda,\ w_T(t,\cdot)\rangle = \langle\lambda,\ w(t,\cdot)\rangle$, *for* $t \geqslant 0$.

Proof. The mild form of (10) is

$$w(t,x) = tp_t(x) - \int_0^t P_{t-s} w^2(s,\cdot)(x)\mathrm{d}s. \tag{12}$$

Then from (11) and (12), we have

$$\langle\lambda, w_T(t,\cdot)\rangle = \int_0^t \langle\lambda, T^2 v_T(Ts,\ T^{1/2}\cdot)\rangle\mathrm{d}s - \int_0^t \langle\lambda,\ w_T^2(s,\cdot)\rangle\mathrm{d}s, \tag{13}$$

and

$$\langle \lambda, w(t, \cdot) \rangle = t - \int_0^t \langle \lambda, w^2(s, \cdot) \rangle ds. \tag{14}$$

By Lemma 1 and Lemma 2, the two terms in the right side of (13) converge to that of (14) respectively and, the desired conclusion is reached.

Proof of Theorem 1. The Laplace functional of $T^{-1}X_T$ is

$$Q\exp(-T^{-1}\langle X_T^{\varrho}, f \rangle) = \exp(-\langle \lambda, v_T(T, \cdot) \rangle - \langle \lambda, u_T(T, \cdot) \rangle ds), \quad f \in C_p^+(\mathbb{R}^d), \tag{15}$$

where $v_T(\cdot, \cdot)$ is the mild solution of (7) and $u_T(\cdot, \cdot)$ is the mild solution of the equation

$$\begin{cases} \dot{u}_T(t) = \Delta u_T(t) - u_T^2(t) + v_T(t), \\ u(0) = 0. \end{cases} \tag{16}$$

Define w_T by $w_T(t, X) := Tu_T(Tt, T^{1/2}x)$, then from (16), w_T satisfies

$$\begin{cases} \dot{w}_T(t,x) = \Delta w_T(t,x) - w_T^2(t) + T^2 v_T(Tt, T^{1/2}x), \\ w_T(0) = 0, \end{cases} \tag{17}$$

and

$$\langle \lambda, w_T(1, \cdot) \rangle = \langle \lambda, u(T, \cdot) \rangle. \tag{18}$$

In the case $\langle \lambda, f \rangle = 1$, by Lemma 2, the limit $w(t,x) := \lim_{T\to\infty} w_T(t,x)$ exists and is the unique positive mild solution of the equation

$$\begin{cases} \dot{w}(t,x) = \Delta w(t,x) - w^2(t,x) + p_t(x), \\ w(0) = 0. \end{cases} \tag{19}$$

For the unnormalised case, we can replace f with θg, where $\theta > 0$ and $\langle \lambda, g \rangle = 1$, and arrive at

$$\begin{cases} \dot{w}(t,x) = \Delta w(t,x) - w^2(t,x) + \theta p_t(x), \\ w(0) = 0. \end{cases} \tag{20}$$

On the other hand, from the mild form of (7), we have

$$\langle \lambda, v_T(T, \cdot) \rangle \leqslant \langle \lambda, P_T(T^{-1}f) \rangle \to 0, \tag{21}$$

as $T \to \infty$. Combining (18), (21) and Lemma 3 with (15), ones obtain that

$$\lim_{T\to\infty} Q\exp(-T^{-1}\langle X_T^{\varrho}, f \rangle) = \exp(-\langle \lambda, w(1, \cdot; \theta) \rangle), \tag{22}$$

where $w(t, x; \theta)$ is given by (20). So we get a limit of $T^{-1}X_T$ as a random positive functional on $C_c(\mathbb{R}^2)^+$, $I(f)$, say, whose Laplace functional is given by (22). By the extension Riesz representation theorem[6], there is a unique random measure μ on $\mathbb{R}^2$, such that $I(f) = \langle \mu, f \rangle = \xi \langle \lambda, f \rangle$, where ξ is a non-negative random variable whose Laplace functional is given by (5). The infinite divisible property of ξ can be derived from its Laplace functional (see ref. [5]).

Furthermore, we have

Theorem 2. *Let $d = 2$, then as $T \to \infty$*

$$T^{-1} X^{\rho}_{Tt} \to \xi_t \cdot \lambda \quad \textit{weakly},$$

where ξ_t is a non-negative increase stochastic process such that

$$Q\exp\{-\theta \xi_t\} = \exp\{-\langle \lambda, w(t, \cdot; \theta) \rangle\},$$

where $w(t, x; \theta)$ is the same as Theorem 1.

Combining Lemma 2, the proof is similar to theorem 3 in ref. [5], the detail is ommited.

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